Analysis 2, Summer 2024

List 5

Review for Exam 1

123. Describe the top half of the circle $x^2 + y^2 = 12$ using parametric equations (or a single vector equation $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath}$) and a range of t values.

There are several correct answer, including $\boxed{\vec{r} = \begin{bmatrix}\sqrt{12}\cos t\\ \sqrt{12}\sin t\end{bmatrix}, 0 \le t \le \pi}$.

124. Calculate $\displaystyle\int_C \cos\left(\tfrac{\pi y^2}{x}\right) \mathrm{d}s$, where C is the line segment from $(0,0)$ to $(6,1)$.

Using $\vec{r} = \begin{bmatrix}6t\\ t\end{bmatrix}$, this is $\displaystyle\int_0^1 \cos\left(\tfrac{\pi (t)^2}{(6t)}\right)\sqrt{(1)^2 + (6)^2}\,\mathrm{d}t = \sqrt{37}\int_0^1 \cos\left(\tfrac{\pi}{6}t\right)\mathrm{d}t = \boxed{\frac{3\sqrt{37}}{\pi}}$

125. Calculate $\displaystyle\iint_D e^{xy}\,\mathrm{d}A$ where $D = \{(x,y) : 1 \le y \le 8, 0 \le x \le \tfrac{1}{y}\}$. $\displaystyle\int_1^8 \int_0^{1/y} e^{xy}\,\mathrm{d}x\,\mathrm{d}y$

Inside: $\displaystyle\int_0^{1/y} e^{xy}\,\mathrm{d}x = \frac{e^{xy}}{y}\Bigg|_{x=0}^{x=1/y} = \frac{e^{x(1/x)}}{y} - \frac{e^0}{y} = \frac{e-1}{y}$.

Outside: $\displaystyle\int_1^8 \frac{e-1}{y}\,\mathrm{d}y = (e-1)(\ln y)\Bigg|_{y=1}^{y=8} = \boxed{(e-1)\ln(8)}$

126. Calculate $f'_{\hat{u}}(0,2)$ where $f(x,y) = \sin(x^2 + \pi y)$ and $\hat{u}$ is parallel to $\vec{v} = \begin{bmatrix}\sqrt{17}\\ 8\end{bmatrix}$.

$|\vec{v}| = 9$, so $\hat{u} = [\frac{\sqrt{17}}{9}, \frac{8}{9}]$ and $f'_{\hat{u}}(0,2) = [0,\pi]\cdot[\frac{\sqrt{17}}{9}, \frac{8}{9}] = \boxed{\frac{8}{9}\pi}$

127. Find the critical point(s) of $f(x,y) = x^2y - 5x^2 - 4xy + 20x$.

$$\begin{cases} 2xy - 10x - 4y + 20 = 0 \\ x^2 - 4x = 0 \end{cases}$$

From the second equation, $x = 0$ or $x = 4$. If $x = 0$ then the first equation becomes $2(0)y - 10(0) - 4y + 20 = 0$, or just $20 - 4y = 0$, so $y = 5$. One CP is $\boxed{(0,5)}$. If $x = 4$ then the second equation becomes $2(4)y - 10(4) - 4y + 20 = 20 - 4y = 0$, so $y = 5$ and the other critical point is $\boxed{(4,5)}$.

128. Find and classify the critical point(s) of $f(x,y) = \ln(-x/y) + ye^x$.

$$\begin{cases} \frac{1}{x} + e^x y = 0 & ① \\ \frac{-1}{y} + e^x = 0 & ② \end{cases}$$

From ②, $y = e^{-x}$. With this, ① becomes $\frac{1}{x} + (e^x)(e^{-x}) = 0$, so $\frac{1}{x} + 1 = 0$, so $x = -1$. Then $y = e^{-(-1)} = e$. The only CP is $(-1, e)$.

$\mathbf{H}f = \begin{bmatrix} e^x y - \frac{1}{x^2} & e^x \\ e^x & \frac{1}{y^2}\end{bmatrix}$. Thus $D(-1,e) = \frac{-1}{e^2} < 0$, so $\boxed{(-1,e) \text{ is a saddle}}$.

129. Find the unit vector $\hat{u} = [u_1, u_2, u_3]$ such that the rate of change of

$$f(x,y,z) = xz^2 - 11\sin(y) + x$$

at $(5,0,1)$ is as large as possible in the direction $\hat{u}$.

Same direction as $\nabla f(5,0,1)$ because $f'_{\hat{u}} = \hat{u}\cdot\nabla f = |\nabla f|\cos\theta$ is largest when the angle between $\hat{u}$ and ∇ is 0. $\nabla f = [1+z^2, -11\cos y, 2xz]$, so $\nabla f(5,0,1) = [2,-11,10]$. This vector has magnitude 15, so $\vec{u} = \frac{1}{15}\nabla f(5,0,1) = \boxed{[\frac{2}{15}, -\frac{11}{15}, \frac{2}{3}]}$.

130. Find and classify the critical point(s) of $f(x,y) = x^2 - 10x + 13 + 4y + y^2$.

$\boxed{(5,-2) \text{ is a local min}}$

131. For $f(x,y) = x\ln(xy)$ at the point $(3, \frac{1}{3})$, $\nabla f(3,\frac{1}{3}) = [1,9]$

(a) in what direction(s) is $f'_{\hat{u}}(3,\frac{1}{3})$ as large as possible?

$\boxed{\text{parallel to } [1,9]}$. As a unit vector, this is $\hat{u} = [\frac{1}{\sqrt{82}}, \frac{9}{\sqrt{82}}]$.

(b) what is the value of $f'_{\hat{u}}(3,\frac{1}{3})$ for the direction from part (a)? $\boxed{\sqrt{82}}$

(c) in what direction(s) is $f'_{\hat{u}}(3,\frac{1}{3})$ as small (most negative) as possible?

$\boxed{\text{parallel to } [-1,-9]}$. As a unit vector, this is $\hat{u} = [\frac{-1}{\sqrt{82}}, \frac{-9}{\sqrt{82}}]$.

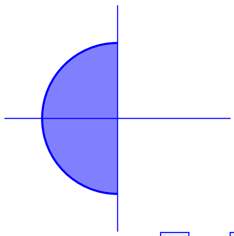
(d) what is the value of $f'_{\hat{u}}(3,\frac{1}{3})$ for the direction from part (c)? $\boxed{-\sqrt{82}}$

(e) in what direction(s) is $f'_{\hat{u}}(3,\frac{1}{3})$ equal to zero?

$\boxed{\text{parallel to } [-9,1]}$. As a unit vector, this is $\hat{u} = [\frac{-9}{\sqrt{82}}, \frac{1}{\sqrt{82}}]$ or $\hat{u} = [\frac{9}{\sqrt{82}}, \frac{-1}{\sqrt{82}}]$.

132. Let D be the region $\{(x,y) : -\sqrt{1-y^2} \le x \le 0\}$.

(a) Draw this region.



(b) Fill in the blanks $\iint_D f\,\mathrm{d}A = \int_{\boxed{-1}}^{\boxed{1}} \int_{\boxed{-\sqrt{1-y^2}}}^{\boxed{0}} f\,\mathrm{d}x\,\mathrm{d}y.$

(c) Fill in the blanks $\iint_D f\,\mathrm{d}A = \int_{\boxed{-1}}^{\boxed{0}} \int_{\boxed{-\sqrt{1-x^2}}}^{\boxed{\sqrt{1-x^2}}} f\,\mathrm{d}y\,\mathrm{d}x.$

☆(d) Fill in the blanks $\iint_D f\,\mathrm{d}A = \int_{\boxed{\pi/2}}^{\boxed{3\pi/2}} \int_{\boxed{0}}^{\boxed{1}} f\,r\,\mathrm{d}r\,\mathrm{d}\theta.$

☆(e) Calculate $\iint_D \frac{e^{x^2+y^2}}{\pi}\,\mathrm{d}A = \int_0^{\pi/2}\int_0^1 \frac{e^{r^2}}{\pi} r\,\mathrm{d}r\,\mathrm{d}\theta = \frac{\pi}{2}\int_0^1 \frac{e^u}{\pi}\,\mathrm{d}u = \boxed{\frac{e-1}{2}}$

Double integrals in "polar coordinates" are not part of MAT 1510, so you will not need to do 132(d) or 132(e) on quizzes or exams in this course.

133. Let R be the triangle with vertices $(0,0)$, $(-6,6)$, and $(6,6)$, and let the density within this triangle be $\rho(x,y) = y+1$.

(a) Evaluate $\iint_R (y+1)\,\mathrm{d}A$. $\boxed{180}$ (This is the mass of the triangle.)

(b) Evaluate $\iint_R (y+1)x \, \mathrm{d}A$. $\boxed{0}$

(c) Evaluate $\iint_R (y+1)y \, \mathrm{d}A$. $\boxed{792}$

(d) The center of mass of the triangle has coordinates $\left(\frac{\text{answer (b)}}{\text{answer (a)}}, \frac{\text{answer (c)}}{\text{answer (a)}}\right)$. Find this point. $\boxed{(0, 4.4)}$

134. For the function $f(x,y) = y\ln(x)$,

(a) Give the gradient vector ∇f at the point $(3,6)$. $\boxed{\begin{bmatrix} 2 \\ \ln 3 \end{bmatrix}}$ or $2\hat{\imath} + \ln(3)\hat{\jmath}$

(b) Give the Hessian matrix $\mathbf{H}f$ at the point $(3,6)$. $\boxed{\begin{bmatrix} -2/3 & 1/3 \\ 1/3 & 0 \end{bmatrix}}$

135. Give all (three) of the first partial derivatives and all (nine) of the second partial derivatives of

$$f(x,y,z) = z^2\ln(xy) + \cos(xz).$$

$f'_x = z^2/x - z\sin(xz)$
$f'_y = z^2/y$
$f'_z = 2z\ln(xy) - x\sin(xz)$

$f''_{xx} = -\frac{z^2}{x^2} - z^2\cos(xz)$
$f''_{xy} = f''_{yx} = 0$
$f''_{xz} = f''_{zx} = 2z/x - \sin(xz) - xz\cos(xz)$
$f''_{yy} = -z^2/y^2$
$f''_{yz} = f''_{zy} = 2z/y$
$f''_{zz} = 2\ln(xy) - x^2\cos(xz)$

136. If $\nabla f(x,y,z) = (3x^3 + z)\hat{\imath} + ze^{yz}\hat{\jmath} + (x + ye^{yz})\hat{k}$, calculate $f''_{xx}(2, 20, -5)$.

$f'_x = 3x^3 + z$, so $f''_{xx} = 9x^2$, and $f''_{xx}(2, 20, -5) = 9(2)^2 = \boxed{36}$.

137. If $\nabla g(x,y) = \begin{bmatrix} \sin(x) \\ \sin(y) \end{bmatrix}$, determine whether $(0,0)$ is a local minimum, local maximum, or saddle point of $g(x,y)$.

$g'_x = \sin(x)$ and $g'_y = \sin(y)$, so $\mathbf{H}g = \begin{bmatrix} \cos(x) & 0 \\ 0 & \cos(y) \end{bmatrix}$ and $D = \cos(x)\cos(y)$. Since $D(0,0) = 1$ and $f''_{xx}(0,0) = 1$, the point $(0,0)$ is a $\boxed{\text{local minimum}}$.

138. For the function $f(x,y) = xy^2$ at the point $(2,3)$,

(a) calculate the directional derivative $f'_{\hat{u}}(2,3)$ in the direction $\hat{u} = [0,1]$. $\boxed{12}$

(b) calculate $f'_{\hat{u}}(2,3)$ when the angle between $\hat{u}$ and $\nabla f(2,3)$ is $60°$. $\boxed{\frac{15}{2}}$

(c) give a formula using θ for the value of $f'_{\hat{u}}(2,3)$ when the angle between $\hat{u}$ and $\nabla f(2,3)$ is θ. $\boxed{15\cos(\theta)}$

(d) what is the largest possible value of $f'_{\hat{u}}(2,3)$, and for what unit vector $\hat{u}$ does this occur? $\boxed{15}$ when $\hat{u} = \boxed{[\frac{3}{5}, \frac{4}{5}]}$

(e) what is the most negative possible value of $f'_{\hat{u}}(2,3)$, and for what unit vector $\hat{u}$ does this occur? $\boxed{-15}$ when $\hat{u} = \boxed{[-\frac{3}{5}, -\frac{4}{5}]}$

(f) give two unit vectors $\hat{u}$ for which $f'_{\hat{u}}(2,3) = 0$. $\boxed{[-\frac{4}{5}, \frac{3}{5}] \text{ and } [\frac{4}{5}, -\frac{3}{5}]}$

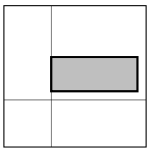
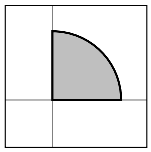
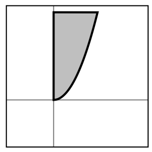
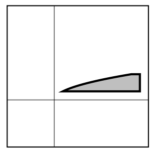
139. Write the system of equations that would be used to find the critical point(s) of

$$f(x,y) = x\sin(xy^3).$$

(Do *not* attempt to solve the system.)

$$\boxed{\begin{cases} \sin(xy^3) + xy^3\cos(xy^3) = 0 \\ 3x^2y^2\cos(xy^3) = 0 \end{cases}}$$

140. Which region below corresponds to $\displaystyle\int_1^3 \int_{y^2}^{10} x^2 \, dx dy$?

(A)

(B)

(C)

$\boxed{\text{(D)}}$


141. Which region above corresponds to $\displaystyle\int_1^3 \int_{y^2}^{10} \frac{x}{y} \, dx dy$? SAME AS PREVIOUS!

142. For the function

$$f(x,y) = x^3 - y^x,$$

give a unit vector $\hat{u}$ for which $f'_{\hat{u}}(3,1) = 0$. A unit vector perpendicular to $\nabla f(3,1) = [6,-3]$ is $\boxed{\frac{2}{\sqrt{5}}\hat{\imath} - \frac{1}{\sqrt{5}}\hat{\jmath}}$ or $\boxed{\frac{-2}{\sqrt{5}}\hat{\imath} + \frac{1}{\sqrt{5}}\hat{\jmath}}$.

143. Calculate $f'_{-\hat{\jmath}}(3,7)$ for $f(x,y) = \dfrac{y}{\cos(x^x)}$. Since $\hat{u} = -\hat{\jmath} = [0,-1]$, this will be

$$\begin{bmatrix} 0 \\ -1 \end{bmatrix} \cdot \nabla f(3,7) \quad = \quad (0)f'_x(3,7) + (-1)f'_y(3,7) \quad = \quad -f'_y(3,7),$$

and we don't need to know f'_x at all. Using $f'_y(x,y) = \dfrac{1}{\cos(x^x)}$ we get $f'_y(3,7) = \dfrac{1}{\cos(3^3)} = \dfrac{1}{\cos(27)}$, and so $f'_{-\hat{\jmath}}(3,7) = \boxed{\dfrac{-1}{\cos(27)}}$.

P.S. If you are curious, $f'_x = x^x y(\ln(x)+1)\dfrac{\sin(x^x)}{(\cos(x^x))^2}$.

144. If $f(x,y)$ is a function for which

$$\begin{array}{lll} f(9,-1)=5 & f(4,7)=6 & f(8,0)=10 \\ f'_x(9,-1)=0 & f'_x(4,7)=0 & f'_x(8,0)=0 \\ f'_y(9,-1)=0 & f'_y(4,7)=1 & f'_y(8,0)=0 \\ f''_{xx}(9,-1)=\frac{1}{2} & f''_{xx}(4,7)=2 & f''_{xx}(8,0)=-6 \\ f''_{xy}(9,-1)=\sqrt{2} & f''_{xy}(4,7)=18 & f''_{xy}(8,0)=0 \\ f''_{yy}(9,-1)=12 & f''_{yy}(4,7)=3 & f''_{yy}(8,0)=-3 \end{array}$$

(a) is $(9,-1)$ a local minimum, local maximum, saddle, or none of these?
$\nabla f(9,-1)=[0,0]$, so $(9,-1)$ is critical point. $D(9,-1)=12\cdot\frac{1}{2}-(\sqrt{5})^2=6-2=4$ is positive, and $f''_{xx}(9,-1)=\frac{1}{2}$ is positive, so $\boxed{(9,-1) \text{ is a local min}}$.

(b) is $(4,7)$ a local minimum, local maximum, saddle, or none of these?
$\nabla f(4,7)=[0,1]\neq[0,0]$, so $(4,7)$ is not a critical point and therefore $\boxed{\text{none}}$ of local min, local max, saddle.

(c) is $(8,0)$ a local minimum, local maximum, saddle, or none of these?
$\nabla f=[0,0]$ and $D=(-6)(-3)-(0)^2=18$ at this point, so it's a $\boxed{\text{local min}}$.

(d) give a vector that is perpendicular to the level curve $f(x,y)=6$ at the point $(4,7)$.
$\nabla f(4,7)=[0,1]$ and gradient vectors are always perpendicular to level curves, so $\hat{u}=\boxed{[1,0]}$. It is also correct to use $\hat{u}=[-1,0]$.

145. Give the Hessian $\mathbf{H}f(x,y)$ of the function $f(x,y)$ for which $\nabla f=\begin{bmatrix} e^{xy}(xy+1) \\ x^2e^{xy} \end{bmatrix}$.

$$\boxed{\begin{bmatrix} ye^{xy}(xy+2) & xe^{xy}(xy+2) \\ xe^{xy}(xy+2) & x^3e^{xy} \end{bmatrix}}$$

☆146. Give an example of a function $f(x,y)$ for which $\nabla f=\begin{bmatrix} 2x+y^2e^{xy^2} \\ 2xye^{xy^2}+9y^2 \end{bmatrix}$.

This will be some

$$f=x^2+e^{xy^2}+g(y),$$

where $g(y)$ is a function of y but is constant with respect to x (and so does not affect f'_x at all). For the function $f=x^2+e^{xy^2}+g$, we have

$$f'_y=0+e^{xy^2}(2yx)+g',$$

so in order to match ∇f from the task it must be that $g'=9y^2$. Therefore $g=3y^3+C$, and

$$\boxed{f=x^2+e^{xy^2}+3y^3+C}$$

describes all functions with the given gradient.

147. Re-write

$$\int_0^{\sqrt{3}} \int_x^{\sqrt{3}x} \frac{y}{x^3 + y^2 x} \, dy \, dx \; + \int_{\sqrt{3}}^{3} \int_x^{3} \frac{y}{x^3 + y^2 x} \, dy \, dx$$

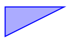
as a single iterated integral.

$$\boxed{\int_0^3 \int_{y/\sqrt{3}}^{y} \frac{y}{x^3 + y^2 x} \, dx \, dy}$$

148. Find the value of $\displaystyle\iint_D \frac{25x^4}{y} \, dA$ where D is the triangle with vertices $(1,1)$ and $(4,4)$ and $(1,4)$.

This can be done as either $\displaystyle\int_1^4 \int_1^y \frac{25x^4}{y} \, dx \, dy = \int_1^4 \left(5y^4 - \frac{5}{y}\right) dy = \boxed{1023 - 10\ln 2}$ or $\displaystyle\int_1^4 \int_x^4 \frac{25x^4}{y} \, dy \, dx = \int_1^4 25x^4(\ln 4 - \ln x) \, dx = \boxed{1023 - 10\ln 2}$. The second version $(dy\, dx)$ requires integration by parts.

149. Evaluate $\displaystyle\int_0^6 \int_{x/2}^3 \sin(\pi y^2) \, dy \, dx$ by reversing the order of integration (that is, by changing to an equivalent integral $dx\, dy$).

The region is a triangle  with $x = 0$ on the left, $y = 3$ on top, and $y = x/2$ (which is also $x = 2y$) as the other side.

$$\int_0^3 \int_0^{2y} \sin(\pi y^2) \, dx \, dy = \int_0^3 2y \sin(\pi y^2) \, dy = \int_0^{9\pi} \tfrac{1}{\pi} \sin(u) \, du = \boxed{\frac{2}{\pi}}$$